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BUCKLING AND POSTBUCKLING OF A LONG, HANGING COLUMN LOWERED INTO A FLUID

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Abstract—A long, hanging column is lowered into a fluid. Depending on the immersion length, densities and flexural rigidity, the column may or may not buckle. The stability criteria are found in terms of Bessel and Airy functions. There exists a maximum stable penetration depth. Postbuckling shapes are also integrated.

1. INTRODUCTION

The buckling of a long, hanging heavy column due to a bottom compressive load is important in the design of drill pipes and risers in offshore oil recovery. For a concentrated vertical bottom force, such as that experienced by a long column lowered onto a rigid surface, Willers (1941) found the buckling load to be 1.0188 $(EI)^{1/3}\rho^{2/3}$ where EI is the flexural rigidity and ρ is the mean density of the column. The nonlinear postbuckling deflections are also integrated (Wang, 1983). Other references on buckling affected by self-weight were given by Wang (1986).

The present paper investigates a different situation, i.e. a long, hanging column slowly lowered into a fluid such as water or mud. Instead of a concentrated bottom force, the compressive load is replaced by a distributed force which may or may not cause buckling. We assume equilibrium shear stresses are negligible in comparison to normal stresses, i.e. steady resistive forces are mainly due to buoyancy.

2. FORMULATION

The origin of the Cartesian system (x', y') is placed at the bottom end of the column (Fig. 1). Let s' be the arc length from the origin and θ be the local angle to the vertical. Let ρ be the weight per length of the column and σ be the buoyancy force per length of the dense medium. The column can be separated into two parts: the bottom part of length l which penetrates into the fluid and the infinite top part above the fluid. Equilibrium of local moments about an elemental length gives the governing equations for the bottom segment,

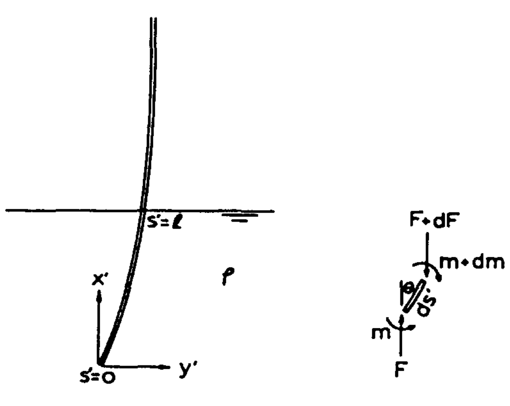

Fig. 1. The coordinate system.

$$dm+s'(\sigma-\rho)\,ds'\sin\theta = 0, \quad 0 \leqslant s' \leqslant l, \tag{1}$$

and for the top segment

$$dm+(l\sigma-\rho s')\,ds'\sin\theta = 0, \quad l \leqslant s' < \infty . \tag{2}$$

Assume the column is slender enough such that local moment is proportional to local curvature,

$$m = EI\frac{d\theta}{ds'}. \tag{3}$$

Normalize all lengths with respect to l and drop primes. The governing equations become

$$\frac{d^2\theta}{ds^2}+K^3 s(1-\lambda)\sin\theta = 0, \quad 0 \leqslant s \leqslant 1, \tag{4}$$

$$\frac{d^2\theta}{ds^2}+K^3(1-s\lambda)\sin\theta = 1, \quad 0 \leqslant s \leqslant \infty. \tag{5}$$

Here the density ratio $\lambda \equiv \rho/\sigma < 1$. The nondimensional parameter $K \equiv l(\sigma/EI)^{1/3}$ represents the relative importance of buoyancy to flexural rigidity. The boundary conditions are that the column is vertical at infinity and that the bottom end is moment free:

$$\theta(\infty) = 0, \tag{6}$$

$$\frac{d\theta}{ds}(0) = 0. \tag{7}$$

Also, angles and curvatures must match at $s = 1$.

3. STABILITY

Equations (4) and (5) are linearized for stability analyses:

$$\frac{d^2\theta}{ds^2}+K^3 s(1-\lambda)\theta = 0, \quad 0 \leqslant s \leqslant 1, \tag{8}$$

$$\frac{d^2\theta}{ds^2}+K^3(1-s\lambda)\theta = 0, \quad 1 \leqslant s < \infty. \tag{9}$$

Let

$$q \equiv K(1-\lambda)^{1/3}s. \tag{10}$$

Equation (8) becomes the Stokes equation:

$$\frac{d^2\theta}{dq^2}+q\theta = 0, \tag{11}$$

with the general solution in terms of Bessel functions:

$$\theta = c_1 q^{1/2} J_{-1/3}(\tfrac{2}{3}q^{3/2}) + c_2 q^{1/2} J_{1/3}(\tfrac{2}{3}q^{3/2}). \tag{12}$$

The constant c_2 is zero due to eqn (7). Let

$$r \equiv K\lambda^{1/3}\left(s - \frac{1}{\lambda}\right). \tag{13}$$

Equation (9) becomes

$$\frac{d^2\theta}{dr^2} - r\theta = 0. \tag{14}$$

The solutions are Airy functions $A_i(r)$, $B_i(r)$. Using eqn (6) we find

$$\theta = c_3 A_i(r). \tag{15}$$

At $s = 1$ both θ and $d\theta/ds$ for the two segments are matched:

$$c_2 q_0^{1/2} J_{-1/3}(\tfrac{2}{3}q_0^{3/2}) = c_3 A_i(r_0), \tag{16}$$

$$-c_2 q_0^{1/2} J_{2/3}(\tfrac{2}{3}q_0^{3/2}) K(1-\lambda)^{1/3} = c_3 A_i'(r_0) K\lambda^{1/3}, \tag{17}$$

where

$$q_0 \equiv K(1-\lambda)^{1/3} > 0, \quad r_0 \equiv K(\lambda^{1/3} - \lambda^{-2/3}) < 0. \tag{18}$$

Since r_0 is negative we use the identities (Abramowitz and Stegun, 1965):

$$A_i(r_0) = \tfrac{1}{3}\sqrt{-r_0}[J_{-1/3}(\zeta_0) + J_{-1/3}(\zeta_0)] \tag{19}$$

$$A_i'(r_0) = \frac{r_0}{3}[J_{-2/3}(\zeta_0) - J_{2/3}(\zeta_0)] \tag{20}$$

where

$$\zeta_0 \equiv \tfrac{2}{3}(-r_0)^{3/2}, \quad \xi_0 \equiv \tfrac{2}{3}(q_0)^{3/2}. \tag{21}$$

Thus for nontrivial c_2, c_3 the characteristic equation is

$$\left(\frac{1}{\lambda} - 1\right)^{1/3} q_0^{1/2} J_{2/3}(\xi_0)[J_{1/3}(\zeta_0) + J_{-1/3}(\zeta_0)] - \sqrt{-r_0} J_{-1/3}(\xi_0)[J_{-2/3}(\zeta_0) - J_{2/3}(\zeta_0)] = 0. \tag{22}$$

For a given λ, eqn (22) is solved numerically for the eigenvalues K. The result is shown in

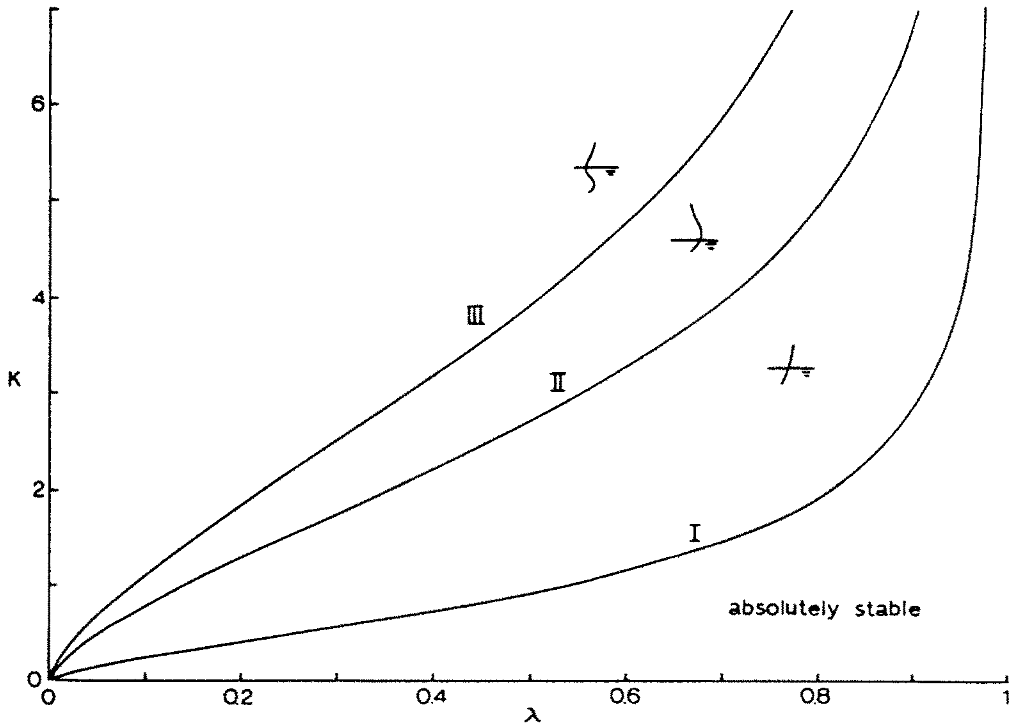

Fig. 2. Eigenvalues K as functions of λ.

Fig. 2 for the first three modes. Any state below curve I is stable, i.e. the column remains straight when lowered into the fluid. Since the value of K increases with the immersed length, the column becomes unstable to the first mode for states above curve I. The maximum K value below which the column is absolutely stable is termed critical K. Above curve II the second mode also becomes unstable. Similar to the case of the Euler column, more nodes may be unstable for larger K.

4. POSTBUCKLING SHAPES

Numerical integration of the nonlinear equations is necessary for finite postbuckling deformations. For given K, λ we guess $\theta(0)$ and integrate eqns (4) and (7) as an initial value problem by the Runge-Kutta-Fehlberg algorithm. At $s = 1$ the resulting values of θ, θ_s are used as initial value for the integration of eqn (5). A solution is found if $\theta \to 0$ asymptotically for large s. If not the value of $\theta(0)$ is adjusted and the procedure is repeated. A step size of $\Delta s = 0.05$ is found to be sufficient for five digit accuracy. "Infinity" is about $s = 6$.

After $\theta(s)$ is determined the postbuckling shapes can be integrated. Since the immersed length l varies for a given column, it is more convenient to normalize x', y' as follows:

$$x = x'(\sigma/EI)^{1/3}, \quad y = y'(\sigma/EI)^{1/3}. \tag{23}$$

Then

$$\frac{dx}{ds} = K\cos\theta, \quad \frac{dy}{ds} = K\sin\theta. \tag{24}$$

Equations (24) are integrated with $x(0) = y(0) = 0$. Figure 3 shows the deformations of the first mode for $\lambda = 0.8$. If $K < 1.920$ or if immersion length $l < 1.92(EI/\sigma)^{1/3}$ then the column is straight. Further lowering causes the column to bend to one side. Figure 4 shows the $\lambda = 0.5$ case. Due to the denser medium, the critical value of K ($=0.9055$) is lower.

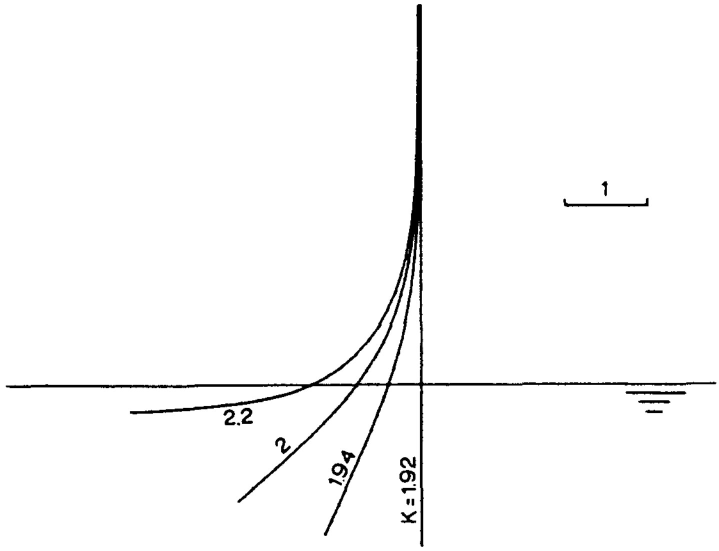

Fig. 3. Postbuckling configurations for $\lambda = 0.8$, I mode. The length unit is $(EI/\sigma)^{1/3}$.

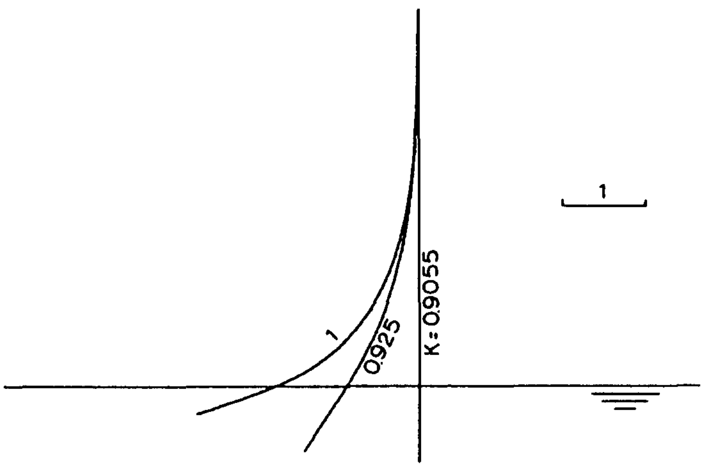

Fig. 4. Postbuckling configurations for $\lambda = 0.5$, I mode.

signifying a much shallower penetration. The first mode has no inflection points. Figure 5 shows the higher modes for $\lambda = 0.5$. The second mode, with one inflection point, is possible if $K > 2.724$. The third mode with two inflection points may occur when $K > 3.908$. As in Euler buckling, the higher modes are seldom realized in practice unless the deformations are physically restrained.

5. DISCUSSION

As an example consider a long steel pipe lowered vertically into a soft sea bed. The pipe, filled with oil, has an outside diameter of 0.5 m and a thickness of 0.02 m. The average density of the pipe is 2423 kg m^{-3}. The densities of sea water and sea bed are 1024 and 2720 kg m^{-3} respectively. Thus $\sigma = (2720-1024)\cdot$ (cross-sectional area) $= 333$ kg m^{-1} and $\rho = 275$ kg m^{-1}, yielding a ratio of $\lambda = 0.825$. From Fig. 2 the critical value of K is 2.1. Using the dimensions we find $EI = 18.4\times 10^{6}$ kg m^{2}. Thus the maximum stable penetration depth is $l = 2.1(EI/\sigma)^{1/3} = 80$ m. Large deformations may occur for immersion lengths greater than 80 m. Figure 3 shows the high sensitivity for K slightly larger than its critical value. Even though a column is initially driven straight into a dense medium, if the

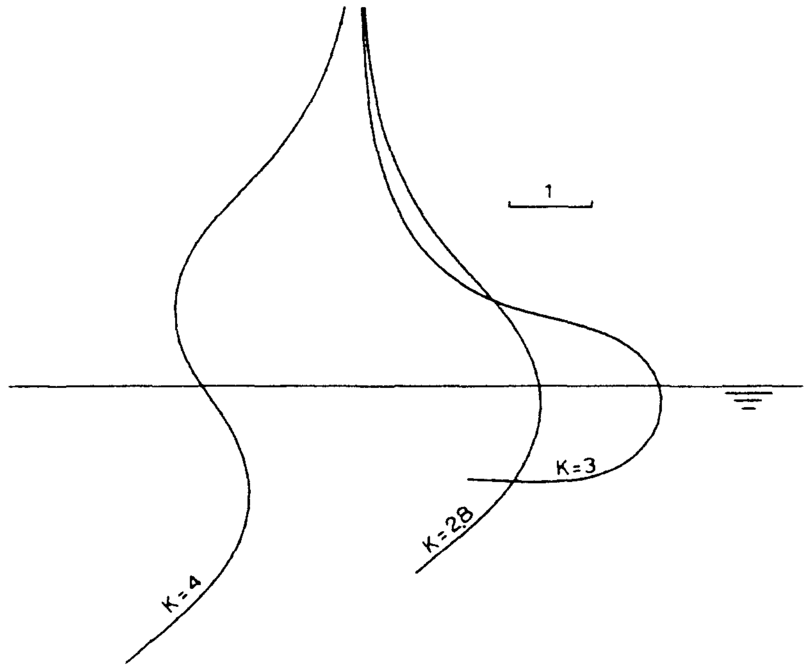

Fig. 5. Postbuckling configurations for $\lambda = 0.5$, right curves II mode, left curve III mode.

value of K is above critical, the column (after slowly overcoming shear resistance) would eventually become curved in shape.

As λ approaches zero, Fig. 2 shows the critical value of K also approaches zero, i.e. no penetration can occur without bending. Under these conditions the normalizations in this paper, using l, become invalid. The dense medium, however, does not approach a solid since it cannot sustain concentrated forces. One is referred to Wang (1983) for the case of lowering a long, hanging column onto an impenetrable solid surface.

REFERENCES

Abramowitz, M. and Stegun, I. A. (1965). (Eds.) *Handbook of Mathematical Functions*. Dover, New York.
Wang, C. Y. (1983). Buckling and postbuckling of a long hanging elastic column due to a bottom load. *J. Appl. Mech.* **50**, 311–314.
Wang, C. Y. (1986). A critical review of the heavy elastica. *Int. J. Mech. Sci.* **28**, 549–559.
Willers, F. A. (1941). Das Knicken Schwerer Gestänge. *Z. Angew. Math. Mech.* **21**, 43–51.